Open Problems

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Nonisomorphic Cospectral Oriented Hypercubes

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Introduction: Let $G^\sigma = (V, \Gamma)$ be any oriented graph obtained by assigning an orientation σ to the edge set E of a simple undirected graph $G = (V, E)$ where $V = \{v_1, v_2, \ldots, v_n\}$. The energy $\mathcal{E}(G)$ of a graph G is the sum of the absolute values of its eigenvalues. This concept was recently generalized to oriented graphs in [1]: if G^σ is an oriented graph and its skew spectrum $\{\mathbf{i}\lambda_1, \mathbf{i}\lambda_2, , \ldots, \mathbf{i}\lambda_n\}$ is the spectrum of its skew adjacency matrix $S(G^\sigma)$ then the skew energy $\mathcal{E}_S(G^\sigma)$ of G^σ is given by $\mathcal{E}_S(G^\sigma) = \sum_{i=1}^{n} |\lambda_i|$.

In [2], Tian has constructed two nonisomorphic orientations of the hypercube Q_d such that the first orientation yields the maximum possible skew energy among all d-regular graphs of order 2^d, namely, $2^d\sqrt{d}$ (see [1]) while in the second orientation, the skew energy equals the energy of the underlying undirected hypercube. We have recently constructed two nonisomorphic orientations of the hypercube Q_d with respect to which the skew energy is equal to the energy of the underlying Q_d.

In regard to this property, we propose the following open problems:

Open Problem 1. Determine the number of nonisomorphic orientations of the hypercube Q_d. Is this already known?

Open Problem 2. Determine the number of nonisomorphic orientations σ of Q_d that yield the maximum skew energy, namely, $2^d\sqrt{d}$.

Open Problem 3. Determine the number of nonisomorphic orientations σ of Q_d for which the skew energy of Q_d^σ is equal to the energy of the underlying Q_d.

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When v is a vertex of a graph/digraph G, the subgraph/subdigraph $G-v$ (in unlabeled form) is called a card of G. The multiset of cards of G is called the deck of G. When G and H are any two graphs or digraphs, vertices $v \in G$ and $w \in H$ are called similar if there is an isomorphism from G to H taking v to w. Deck of G at v is the collection of cards of G corresponding to deletion of vertices other than v, and in each of which v is given color blue. If G has n vertices, then the deck of G at v has $n-1$ cards. Vertices $v \in G$ and $w \in H$ are called subgraph/subdigraph equivalent if the deck of G at v is same as the deck of H at w.

Problem. G and H are any two graphs on at least four vertices. $v \in G$ and $w \in H$ are subgraph equivalent. Are $v \in G$ and $w \in H$ similar?

Ramachandran and Bhanumathy [2] have proved it to be true in some cases including the following:
(i) G is regular.
(ii) One among G and G^C is disconnected with positive degree for v. (G^C denotes the complement of G).
(iii) v is a cut vertex of G or G^C.
(iv) A specific 2-vertex coloring of $G-v$ is reconstructible.
(v) $G-v$ is either disconnected, regular, a tree, unicyclic, or separable without endvertices.

A general conjecture proposed by Harary and Manvel [1] while studying the graph reconstruction conjecture for partially labeled graphs includes as a particular case the conjecture that the above problem is true for all graphs.

No example for which the above problem fails is so far known.

The corresponding problem for digraphs is false. But when the degree pair of the deleted vertex is also given with the cards of G at v, that digraph version of the above problem is open.

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Antipodal graphs

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The antipodal graph $A(G)$ of a given graph G has the same vertex set $V(G)$ and two vertices are adjacent in $A(G)$ if the distance between them in G is equal to the diameter of G.

Known Result:

A graph G is antipodal if and only if $\overline{G}$ is of diameter 2 or $\overline{G}$ is disconnected and every component is complete.

Problem 1. Find graphs G for which $A(G)$ is connected.

Problem 2. Find graphs G for which $A(G)$ is regular.

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The graphs $G(V, E)$ considered here are simple graphs. Let F be a complete graph (K_m), the complement of a complete graph ($\overline{K_m}$), star, complete bipartite graph ($K_{m,n}$), cycle (C_n), path (P_n), tree, triangulated graph, or a graph other than these. Denote

- $\theta_F(G)$ = minimum number of subgraphs F (F-subgraphs) of G needed to cover the vertices of G.
- $\alpha_F(G)$ = maximum number of vertices in G such that no two of them are in the same F .

Obviously, $\alpha_F(G) \leq \theta_F(G)$.

G is F-perfect if $\theta_F(H) = \alpha_F(H)$ for every induced subgraph H of G. If, $F = K_m$ or $\overline{K}_m$, then F-perfectness is Berge's α-perfectness or γ-perfectness.

Problem 1. Given an F, characterize F-perfect graphs.

Problem 2. Characterize those F-perfect graphs G, such that G and its compliment $\overline{G}$ are F-perfect.

Given F, let M-F-set and m-F-set respectively denote an F-maximal independent set with maximum and minimum cardinality among all the F-maximal independent sets of G. An F-independent set in G is F-good if it meets all the maximal F-subgraphs of G. G is F-strongly/M-F-strongly/m-F-strongly perfect if every induced subgraph of G contains an F-independent set/M-F-set/m-F-set which is F-good.

Problem 3. Investigate an m-F-set, for a suitable F.

Problem 4. Given a suitable F, characterize F-strongly/M-F-strongly/(m-F-strongly) perfect graphs.

Note: If we restrict F to an induced subgraph of G, then the F-perfect graphs will have different properties. For example, if F is an induced cycle, tree or path of G, the F-perfectness will enjoy more nontriviality. One could also color G differently using F-independent sets of G.

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A decomposition of a graph G is a partition of G into edge-disjoint subgraphs of G. A cycle passing through all the vertices of a graph is called a hamilton cycle. A 2-factor in a graph G is a 2-regular spanning subgraph of G. A 2-factorization of a graph G is a decomposition of G into 2-factors. A 2-factor in which each component is a C_4 is called a C_4-factor.

Question: For what values of even $n \geq 8$, it is possible to factorize $K_{n,n} - H$ into C_4-factors, where H is a hamilton cycle of $K_{n,n}$?

It is known that the graph $K_{4,4}$ can not be decomposed into a hamilton cycle and a C_4-factor. We have proved that the graph $K_{6,6}$ can not be decomposed into a hamilton cycle and two C_4-factors.

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Some problems on b-coloring of graphs

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A b-coloring of a graph G is a proper coloring of G in which each color class has a color dominating vertex (c.d.v.), that is, a vertex that has a neighbor in each of the other color classes. The b-chromatic number, $b(G)$, of G is the largest k such that G has a b-coloring using k colors. The concept of b-coloring was introduced by Irving and Manlove in analogy to the achromatic number of a graph G (which gives the maximum number of color classes in a complete coloring of G). They have shown that the determination of $b(G)$ is NP-hard for general graphs, but polynomial for trees. From the very definition of $b(G)$, the chromatic number $\chi(G)$ of G is the least k for which G admits a b-coloring using k colors. Thus $\chi(G) \leq b(G) \leq 1 + \Delta(G)$, where $\Delta(G)$ is the maximum degree of G.

The b-spectrum of a graph G, denoted by $S_b(G)$, is defined by:

$$S_b(G) = \{k \ : \ G \text{ has a } b\text{-coloring using } k \text{ colors}\}.$$

Clearly, $S_b(G) \subseteq \{\chi(G), \ldots, b(G)\}$ and G is b-continuous iff $S_b(G) = \{\chi(G), \ldots, b(G)\}$.

Let us recall the definitions of the Cartesian product of graphs, Kneser graphs and Mycielskian of a graph.

The Cartesian product of two graphs $G = (V_1, E_1)$ and $H = (V_2, E_2)$, denoted by $G\Box H$, has vertex set $V_1 \times V_2$, and two vertices (x_1, y_1) and (x_2, y_2) are adjacent in $G\Box H$ iff $x_1 = x_2$ and y_1 is adjacent to y_2 in H, or $y_1 = y_2$ and x_1 is adjacent to x_2 in G.

Let n and k be positive integers, $m = 2n + k$, where $k \geq 1$. We denote by $[m]$ the set $\{1, 2, \ldots, m\}$ and by $\binom{[m]}{n}$ the collection of all n-subsets of $[m]$. The Kneser graph $K(m, n)$ has vertex set $\binom{[m]}{n}$, in which two vertices are adjacent iff the corresponding n-subsets are disjoint. When $k = 1$, we have the odd graphs. The famous Petersen graph is the odd graph $K(5, 2)$.

In a search for triangle-free graphs with arbitrarily large chromatic numbers, Mycielski developed an interesting graph transformation as follows. For a graph $G = (V, E)$, the Mycielskian of G is the graph $\mu(G)$ with vertex set $V \cup V' \cup \{u\}$, where $V' = \{x' : x \in V\}$ and edge set $E \cup \{xy' : xy \in E\} \cup \{y'u : y' \in V'\}$. The vertex x' is called the twin of the vertex x (and x the twin of x') and the vertex u is called the root of $\mu(G)$.

We present below some open problems on b-coloring of graphs; the references contain further technical details.

- Does there exist graph G with $b(\mu(G)) \geq 2b(G)$?
- For which graphs G is $b(G) - 1 \leq b(G - v) \leq b(G)$, for any $v \in V(G)$?
- Is it true that $\mu(G)$ is b-continuous whenever G is b-continuous?
- Find the b-chromatic number of Kneser graphs. (For $KG(m, 2)$ and odd graphs, the b-chromatic number is known)
- Are Kneser graphs b-continuous?
- Characterize graphs G for which $b(G\Box H)$ =max$\{b(G), b(H)\}$, where $\Box$ denotes the cartesian product of G and H. (In general, $b(G\Box H) \geq$ max $\{b(G), b(H)\}$)
- Find trees T_1 and T_2 with $b(T_1\Box T_2) = b(T_1) + b(T_2) - 1$ and $b(T_1) \leq \Delta(T_1)$.
- What is $b(G\Box H)$ for chordal graphs G and H?
- When is $G\Box H$ b-continuous?
- Is it true that any regular graph with girth at least 5 is b-continuous?

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